Indian Institute of Science Education and Research, Tirupati

ABHIPRAJNA 2023 (Prelims)

MATH QUESTION PAPER

Date: September 17, 2023 Maximum points: 30

Instructions:

- *All questions are mandatory*
- *The question paper consists of 8 questions in total, spread over 5 pages.*
- *Answers must be written in legible and readable handwriting, failing which that question shall not be considered for evaluation.*
- ****Question 3 (ii) is the Hint question, whose answer will be required to solve the puzzle question in the Theme Round.***
- *Click this link for submitting your solution PDF:* *https://forms.gle/2csqGoKj3p66rmpo9*

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1. **(3 points)**

   Let's define the following sets:

   $$G := \{g \mid g : \{1,2,3\} \to \mathbb{R}\}$$
   $$P_3 := S = \{p(x) \mid p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \text{ where } a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

   Consider the set $S := \{\mathbb{R}^3, \mathbb{R}^4, G, P\}$.

   For each ordered pair (A, B) with distinct A and B in S, consider the set

   $$F[(A,B)] = \{f : A \to B \,|\, f \text{ is surjective and } f(ax+by) = af(x) + bf(y)$$
   $$\text{where } a, b \in \mathbb{R} \text{ and } f(x) = 0 \text{ iff } x = 0\}.$$

   For each ordered pair (A, B) find if F is nonempty and justify.

2. **(3 points)**

   Let P be a point inside triangle ABC. Let perpendiculars from each vertex A, B, C of the triangle meet the opposite sides at D, E, F respectively. Similarly, construct perpendiculars from P onto each side of the triangle. Let the points of intersection of perpendiculars with sides BC, AC, AB be G, H, I respectively.

   Assume that $\frac{AD}{PG} = \frac{BE}{PH} = \frac{CF}{PI}$. Call this ratio 'r'.

(i) Prove that P is the centroid of Triangle ABC.

(ii) Find the value of the common ratio 'r'.

3. **(3 points) ***

For each prime number p consider the 'p'-faced dice with numbers labeled from 1 to p (unbiased).

(i) Each such dice is thrown twice and observation is noted. (Any two throws of dice are considered independent). Let E denote the event that the number '1' turns up in every throw for every dice. Find the probability of E not occuring.

(ii) Now say, each such dice is thrown only once and observation is noted. Let F denote the event that the number '1' turns up in that one throw for every dice. Find the probability of F not occurring.

4. **(3 points)**

Let $p(x) = x^n + a_1 x^{n-1} + \ldots + a_n = 0$ be a polynomial of degree $n > 1$. Prove that if $2a_1^2 < 5a_2$, then all roots of $p(x)$ cannot be real.

5. **(4 points)**

Define a sequence of polynomials by $P_0(x) = 1$,
$P_n'(x) = P_{n-1}(x)$, and
$\int_0^1 P_n(x)\,dx = 0$ for $n = 1, 2, 3, \ldots$

(i) Show that $P_n(0) = P_n(1)$ for all $n \geq 2$.

(ii) If $P_n(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} p_k x^{n-k}$, then find a general formula for p_n.

(iii) Show that $P_n(1-x) = (-1)^n P_n(x)$.

6. **(4 points)**

In a mystical land called *Numeralia*, there are *Magical lanes* which are spread across infinitely in both directions. The lanes are engraved with digits not equal to 9 ie., from the set {0,1,...,8} . Each possible such engravings correspond to one lane. A car travelling on any of these lanes has the speciality that whenever it crosses digits on the lane the value of its speed gets concatenated by that digit (to the unit place). The car maintains its direction if the successive value of the speed obtained by concatenation is composite and reverses the direction if it is prime (assume the speed values are changing in a discrete manner).

The Car starts with speed S_1 which is a positive integer. Thus a sequence of positive integers ($S_1, S_2, S_3, ...$) is generated. Call it a sequence of speeds.

Part of one such engraving and how the car moves on that lane is illustrated.

Figure 1: **Motion of Car when the sequence of speeds is (34, 345, 3457, 34577, 345773, ...). Arrows represents direction of motion of car.**

Eventually, does the car inevitably travel in one direction indefinitely, or will it continually reverse its direction an infinite number of times? Can both scenarios occur in different lanes ? Justify your deduction.

(Bonus points if you give a reason for excluding the digit 9 from being concatenated and how it would affect the existence of a solution to the above question.)

7. **(4 points)**

In the land of Gridville, a peculiar problem existed. Its residents kept both goats and tigers and had devised a unique way to protect their precious goats from the lurking tigers as well as feed the tigers.

At the heart of Gridville lay a beautiful **n x n** square grid, consisting of n columns and n rows. The farmers placed a goat ('x') and a tiger ('o') in each row and column of the grid so that every row and every column could have exactly one goat and one tiger. Such a placement leads to a "Grid Arrangement". In the hypothetical land of Gridville, Grid Arrangement made sure that the Goats are protected and Tigers were well fed.

The column index of 'x' or 'o' is the column number (numbered from left to right conventionally) at which it is placed in a given row.

Say x_i, x_j are column indices of x's in rows i and j respectively, and o_i, o_j are column indices of the o's in rows i and j respectively. Then rows represented by i and $i+1$ are called 'adjacently-locked' if they satisfy the condition:

$$x_i < x_{i+1} < o_i < o_{i+1} \quad \text{or} \quad x_i < o_{i+1} < o_i < x_{i+1}$$

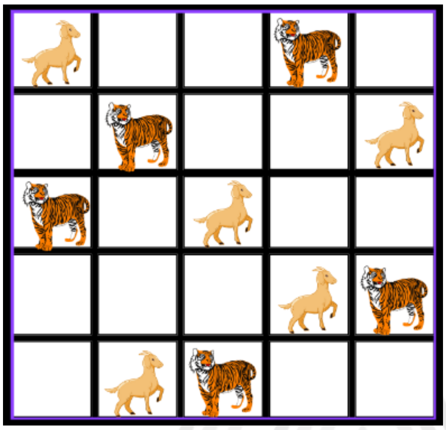

Figure 2: **Grid Figure with $x_1 = 1, x_2 = 5, o_1 = 4, o_2 = 2$. Row 1 and 2 are Adjacently Locked.**

Find the formula for number of Grid Arrangements possible with Rows i and i+1 Adjacently Locked.

8. **(6 points)**

You are presented with a chocolate bar which extends infinitely across the two dimensional plane, and cuts are made through it using lines that extend infinitely in both directions.

Denote by m, the number of lines used to make the cuts and by N the number of pieces obtained in the chocolate partition.

(i) Find the maximum and minimum values of N possible for a fixed m, in terms of m.

The maximum number of parallel lines/cuts in the partition is denoted by a. The maximum number of concurrent lines (lines passing through a single point) in the partition is denoted by b. Denote by r_i $(2 \leq i \leq b)$, the number of lines having exactly i lines passing through them.

(ii) Show that

$$N = m + 1 + \sum_{i=2}^{b} (i-1) r_i.$$

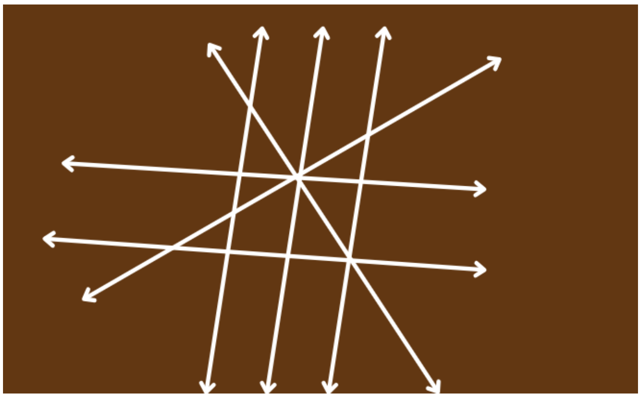

Figure 3: **A partition of the infinite chocolate with 7 cuts where a = 3, b = 4, N = 21.** Note that the figure is just representative and the chocolate actually extends to infinity.

(iii) Show that

$$N \geq (a+1)(m-a+1)$$

(iv) Deduce from the construction or from the bound above (bonus points for doing both) that N cannot belong to the interval $(m+1, 2m)$ for $m \leq 3$.

Click this link for submitting your solution PDF:

https://forms.gle/2csqGoKj3p66rmpo9

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